Mathematical Biology
Selection in a Population of Three Phenotypes

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Introduction

In this study, we examine the effects of selection in a population with three distinct phenotypes. The population consists of moths with three genotypes: WW, Ww, and ww, which correspond to white wing color, gray wing color, and black wing color, respectively. Individuals possessing the WW or ww genotypes are referred to as homozygous, while those with the Ww genotype are considered heterozygous. The wing colors are determined by the W and w alleles, where W is responsible for white wings and w for black wings. When the alleles combine in a heterozygous pair (Ww), gray wings are produced. The study incorporates the concept of selection to explore how these variations impact the population.

Task 1

We need to derive the iterative map for the frequency of the W allele as outlined in equation (2.34) from the course material. The goal is to examine the dynamic behavior of allele frequencies over time, while utilizing a Punnett square to model the inheritance of alleles and develop the recurrence relation $P_{n+1} = f(P_n)$. In addition, we will explain the meaning of the variables and parameters α, β, and γ in relation to the selection process. We begin by computing P_{n+1} via a punnentt square.

Table 1: Punnett square summarizing how the probabilities of the W and w alleles in the current generation predicts future generation's genotypes.

	Mother W (p_n)	**Mother w** $(1 - p_n)$
Father W (p_n)	p_n^2	$p_n(1 - p_n)$
Father w $(1 - p_n)$	$p_n(1 - p_n)$	$(1 - p_n)^2$

Now, the frequencies in the next generation are given by:

$$
\begin{aligned}
&WW : P_n^2 \\
&Ww + wW : 2P_n(1-P_n) \\
&ww : (1-P_n)
\end{aligned}
$$

Also, we need to take notice of the probability of obtaining a W allele given below:

$$
\begin{aligned}
&WW : 1 \\
&Ww : \frac{1}{2} \\
&ww : 0
\end{aligned}
$$

Let:

- $0 \leq \alpha \leq 1$: the fraction of white-winged moths that survive to produce the next generation.
- $0 \leq \beta \leq 1$: the fraction of gray-winged moths that survive to produce the next generation.
- $0 \leq \gamma \leq 1$: the fraction of black-winged moths that survive to produce the next generation.

Then, The future probability, P_{n+1} is given as:

$$
\begin{aligned}
P_{n+1} &= \frac{\text{Number of } W}{\text{Total number of } W + w} \\
&= \frac{\alpha {P_n}^2 + \frac{1}{2}(2)\beta P_n(1-P_n) + 0(1-P_n)^2}{\alpha {P_n}^2 + 2\beta P_n(1-P_n) + \gamma(1-P_n)^2} \\
&= \frac{\alpha {P_n}^2 + \beta P_n(1-P_n)}{\alpha {P_n}^2 + 2\beta P_n(1-P_n) + \gamma(1-P_n)^2} \\
&= \frac{\beta P_n + (\alpha-\beta){P_n}^2}{(\alpha - 2\beta + \gamma){P_n}^2 + 2(\beta-\gamma)P_n + \gamma} = f(P_n)
\end{aligned}
$$

For fixed points, $P_{n+1} = P_n = P$.

Thus,

$$\begin{aligned}
P &= \frac{\beta P + (\alpha-\beta)P^2}{(\alpha-2\beta+\gamma)P^2 + 2(\beta-\gamma)P + \gamma} \\
0 &= \frac{\beta P + (\alpha-\beta)P^2}{(\alpha-2\beta+\gamma)P^2 + 2(\beta-\gamma)P + \gamma} - P\frac{(\alpha-2\beta+\gamma)P^2 + 2(\beta-\gamma)P + \gamma}{(\alpha-2\beta+\gamma)P^2 + 2(\beta-\gamma)P + \gamma} \\
0 &= \frac{P[\beta + (\alpha-\beta)P - (\alpha-2\beta+\gamma)P^2 - 2(\beta-\gamma)P - \gamma]}{(\alpha-2\beta+\gamma)P^2 + 2(\beta-\gamma)P + \gamma}, p \neq 0
\end{aligned}$$

This is the same as writing :

$$P[\beta + (\alpha-\beta)P - (\alpha-2\beta+\gamma)P^2 - 2(\beta-\gamma)P - \gamma] = 0$$

Dividing through by P and grouping like-terms gives:

$$\begin{aligned}
0 &= -(\alpha-2\beta+\gamma)P^2 + (\alpha-\beta-2(\beta-\gamma))P + (\beta-\gamma) \\
&= P^2 - \frac{\alpha-\beta-2(\beta-\gamma)}{\alpha-2\beta+\gamma}P + \frac{\gamma-\beta}{\alpha-2\beta+\gamma}
\end{aligned}$$

We have that, $P_1^* = 0$ is a solution to the equation. This indicates that the W allele will be extinct, leading to the disappearance of the genotypes WW, and Ww. Thus, the white-winged and gray-winged moths will be extinct and as a result, only black-winged moths would survive to produce the next generation.

Again, for $P_2^* = 1$, we have that

$$(1)^2[\beta + (\alpha-\beta)(1) - (\alpha-2\beta+\gamma)1^2 - 2(\beta-\gamma)(1) - \gamma] = \beta + \alpha - \beta - \alpha + 2\beta - \gamma - 2\beta + 2\gamma - \gamma = 0$$

Thus, $P_2^* = 1$ is also a solutions to the equation. This indicates that the w allele will be extinct, leading to the disappearance of the genotypes Ww and ww. Thus, the black-winged and gray-winged moths will be extinct and as a result, only white-winged moths would survive to produce the next generation.

Lastly, we have $P_3^* = \frac{\gamma-\beta}{\alpha-2\beta+\gamma} = \frac{\gamma-\beta}{(\gamma-\beta)+(\alpha-\beta)} = \frac{1}{1+(\frac{\alpha-\beta}{\gamma-\beta})}, P_3^* \in [0,1]$.

The value of P_3^* reflects the possibility of all three genotypes WW, Ww, ww coexisting after selection to produce the next generation. This can only be true when $\beta \geq \alpha \geq \gamma$ or $\beta \geq \gamma \geq \alpha$.

For stability, $P_{n+1} = f(P_n)$

Differentiating $f(P_n)$ yeilds:

$$f'(P_n) = \frac{(2(\alpha-\beta)P_n + \beta)((\alpha-2\beta+\gamma)P_n^2 + 2(\beta-\gamma)P_n + \gamma) - ((\alpha-\beta)P_n^2 + \beta P_n)(2(\alpha-2\beta+\gamma)P_n + 2(\beta-\gamma))}{[(\alpha-2\beta+\gamma)P_n^2 + 2(\beta-\gamma)P_n + \gamma]^2}$$

$$f'(P_n) = \frac{\begin{array}{c}2(\alpha-\beta)(\alpha-2\beta+\gamma)P_n^3 + \beta(\alpha-2\beta+\gamma)P_n^2 + 4(\alpha-\beta)(\beta-\gamma)P_n^2 + 2\beta(\beta-\gamma)P_n \\ +(2\gamma(\alpha-\beta)P_n + \beta\gamma) - 2(\alpha-\beta)(\alpha-2\beta+\gamma)P_n^3 - 2(\alpha-\beta)(\beta-\gamma)P_n^2 - 2\beta(\alpha-2\beta+\gamma)p_n^2 - 2\beta(\beta-\gamma)P_n\end{array}}{[(\alpha-2\beta+\gamma)P_n^2 + 2(\beta-\gamma)P_n + \gamma]^2}$$

Finally, it simplifies further to:

$$f'(P_n) = \frac{(\alpha\beta - 2\alpha\gamma + \beta\gamma)P_n^2 + 2\gamma(\alpha-\gamma)P_n + \beta\gamma}{[(\alpha-2\beta+\gamma)P_n^2 + 2(\beta-\gamma)P_n + \gamma]^2}$$

For $P_1^* = 0$, we have:

$$f'(0) = \frac{(\alpha\beta - 2\alpha\gamma + \beta\gamma)(0) + 2\gamma(\alpha-\gamma)(0) + \beta\gamma}{[(\alpha-2\beta+\gamma)(0) + 2(\beta-\gamma)(0) + \gamma]^2} = \frac{\beta\gamma}{\gamma^2} = \frac{\beta}{\gamma}$$

If

- If $\beta > \gamma$, then $|f'(0)| > 1 \implies P_1^* = 0$ is unstable.
- If $\beta < \gamma$, then $|f'(0)| < 1 \implies P_1^* = 0$ is stable.

For $P_2^* = 1$, we have:

$$\begin{aligned} f'(1) &= \frac{(\alpha\beta - 2\alpha\gamma + \beta\gamma)(1) + 2\gamma(\alpha-\beta)(1) + \beta\gamma}{[(\alpha-2\beta+\gamma)(1)^2 + 2(\beta-\gamma)(1) + \gamma]^2} \\ &= \frac{\alpha\beta - 2\alpha\gamma + \beta\gamma + 2\gamma(\alpha-\beta) + \beta\gamma}{(\alpha-2\beta+\gamma+2\beta-2\gamma+\gamma)^2} = \frac{\alpha\beta}{\alpha^2} = \frac{\beta}{\alpha} \end{aligned}$$

If

- If $\beta > \alpha$, then $|f'(1)| > 1 \implies P_2^* = 0$ is unstable.
- If $\beta < \alpha$, then $|f'(1)| < 1 \implies P_2^* = 0$ is stable.

Also for $P_3^* = \frac{\gamma-\beta}{(\alpha-\beta)+(\gamma-\beta)}$, we have:

$$f'\left(\frac{\gamma-\beta}{(\alpha-\beta)+(\gamma-\beta)}\right) = \frac{(\alpha\beta - 2\alpha\gamma + \beta\gamma)}{\alpha\gamma - \beta^2}$$

For all P_i^*, where $i = 1, 2, 3$, the point P_i^* is stable if $|f'(P_i^*)| < 1$.

Part 2

In this section, we need to analyze the long-term behavior of the W allele frequency equation (2.34) under four distinct cases by using cobweb diagrams to visualize the iterative dynamics. The cobweb diagrams are given below:

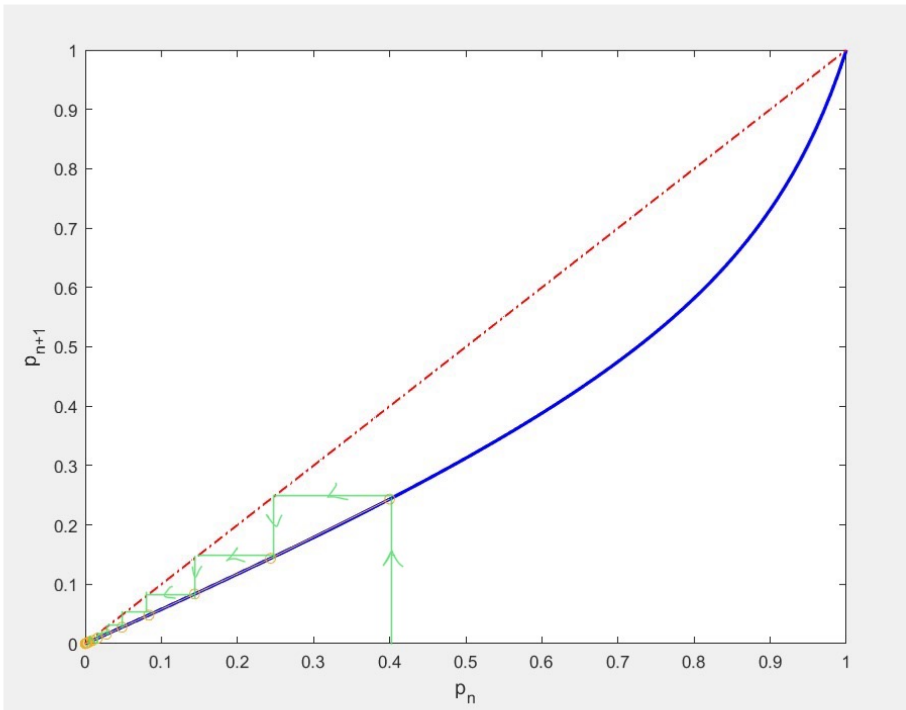

Figure 1: Case (a): converges to 0. Only black-winged moths survives

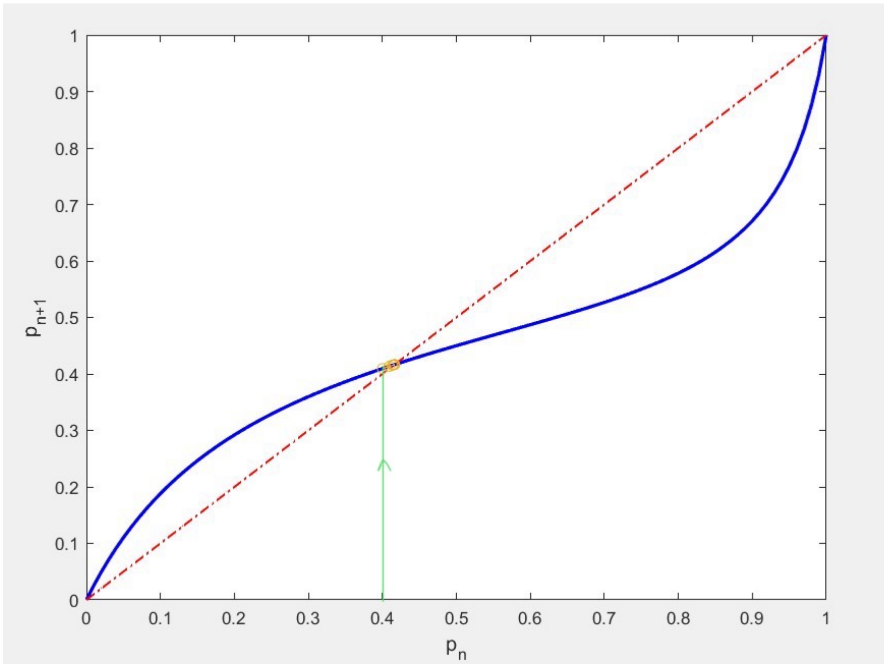

Figure 2: Case (b): converges to $P_3{}^*$. Co-existence, due to the survival of gray-winged moths.

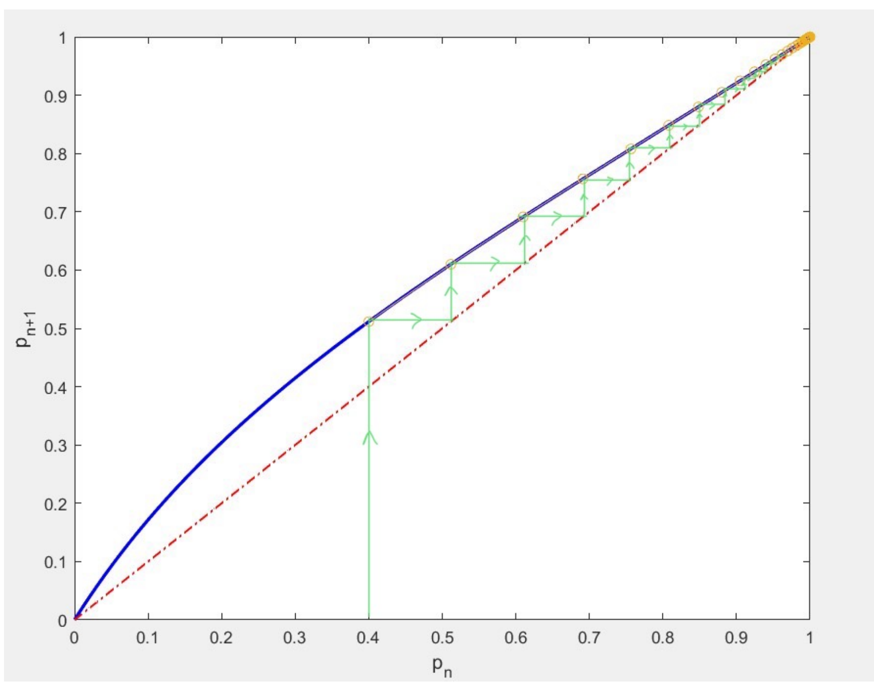

Figure 3: Case (c): converges to 1.Only white-winged moths survives.

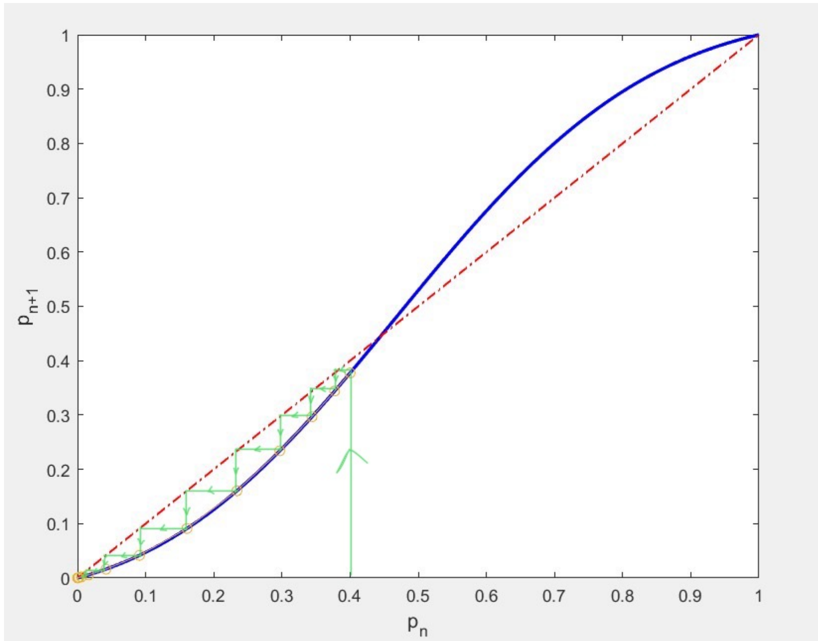

Figure 4: Case (d) with $P_1 = 0.4$.

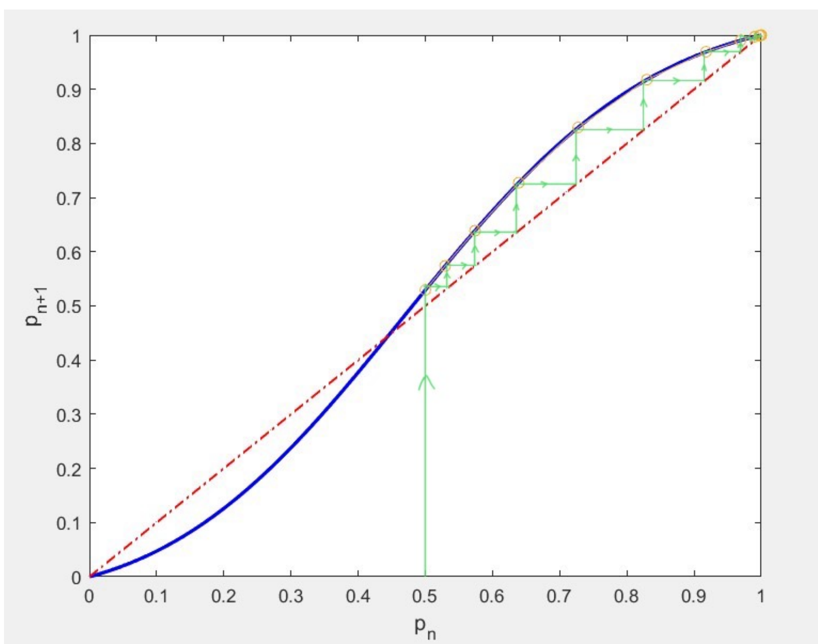

Figure 5: Case (d) with $P_1 = 0.5$. Thus, convergence depends on initial point

From the diagrams:

- Case (b) is the only scenario where all three phenotypes will coexist. This is because the gray-winged moths has a selective advantage over both the white-winged and black-winged moths.

- In cases (a), (c), and (d), the gray-winged moths will be driven extinct. This is because the gray-winged moths has a selective disadvantage and hence won't survive the selection pressure.

- The long-term behaviour will depends on the initial conditions in case (d). If the initial condition is slightly below the point ${P_3}^*$, the white-winged moths will go extinct. Conversely, if the initial condition is slightly above ${P_3}^*$, the black-winged moths will go extinct. In either scenario, gray-winged moths will eventually go extinct as well, as only one of the alleles will survive.

Conclusion

This study explored the effects of selection on a population with three distinct phenotypes—white-winged (*WW*), gray-winged (*Ww*), and black-winged (*ww*) moths—using a dynamic model of allele frequencies. Through iterative mapping and stability analysis, we derived conditions under which the population converges to specific equilibrium states. Our findings demonstrate that the survival of phenotypes depends critically on selection coefficients (α, β, γ) and initial allele frequencies.
The analysis identified three potential fixed points: extinction of either the W or w allele, leading to dominance of a single phenotype, or coexistence of all three phenotypes under specific selection conditions ($\beta \geq \alpha \geq \gamma$ or $\beta \geq \gamma \geq \alpha$). Cobweb diagrams provided visual insights into these dynamics, showing that coexistence is rare and achievable only when gray-winged moths hold a selective advantage. In all other cases, the gray-winged phenotype is driven to extinction.
Additionally, the study revealed the sensitivity of long-term population behavior to initial conditions, particularly in scenarios where equilibrium depends on the relative fitness of alleles. This highlights the importance of selection pressures and initial allele distributions in shaping evolutionary outcomes.